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A Linear Independence Measure for Certain p-Adic Numbers

K. VÄÄNÄNEN*

Department of Mathematics, University of Oulu, 90570 Oulu 57, Finland

AND

R. WALLISSER

Mathematisches Institut der Albert-Ludwigs-Universität, Albertstraße 23, 7800 Freiburg im Breisgau, Germany

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A linear independence measure is obtained for certain values of a p-adic function $\sum_{n=0}^{\infty} q^{n(n-1)/2} z^n$, where $q \in \mathbb{Q}$, $0 < |q|_p < 1$.

1. Introduction

In 1921 Tschakaloff [10] proved the linear independence over $\mathbb{Q}$ of certain values of the function

$$f(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} z^n, \tag{1}$$

where q, $0 < |q| < 1$, is a rational number satisfying certain conditions. In the case $q^{-1} \in \mathbb{Z}$ these results were generalized by Skolem [7] to give the linear independence of the values of the derivatives of f. He used a method following closely certain transcendence proofs of e, see, e.g., Perron [6]. More recently the arithmetic properties of the values of f have been studied

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by several authors; in particular the recent work of Bezivin [2] contains as a special case the above results. Further, quantitative irrationality measures for certain sets of values of f have been obtained by Bundschuh and Shiokawa [3] and Stihl [8]. In [11] we showed that Skolem's method can be refined to obtain irrationality measures for sets containing values of the derivatives of f.

The first studies on arithmetic properties of the p-adic function (1) in the case $q = p^m$, $m \in \mathbb{N}$, were made by Icen [4], who proved that this function cannot have algebraic values at too many algebraic points. The linear independence of certain values of this same p-adic function and its derivatives was obtained by Bezivin [2]. In [11] it was shown that Skolem's method can be used to obtain quantitative estimates also in the p-adic case if q, $0 < |q|_p < 1$, satisfies certain conditions. However, it seems that, e.g., the case $q = p$ cannot be handled by this method. In the present paper our purpose is to obtain a linear independence measure for the values of the p-adic function (1) by using the Padé approximation polynomials of Stihl [8]. In this paper we give a p-adic analogue of the main result of Bundschuh and Shiokawa [3] and of certain results of Stihl [8].

2. Results

We shall use $|\ |$ to denote the usual archimedean absolute value. If p is a prime, then $|\ |_p$ is used to denote the p-adic valuation of $\mathbb{Q}_p$ normalized by $|p|_p = p^{-1}$. Let us suppose that $q = s/r \in \mathbb{Q}$, $(s, r) = 1$, $r > 0$, and $|s| > 1$. If p is a prime factor of s, then $|q|_p < 1$ and the series (1) converges for all $z \in \mathbb{Q}_p$. Our main result is the following linear independence measure for some values of this p-adic function f.

THEOREM 1. *Let $\gamma_1, ..., \gamma_D$ denote non-zero rational numbers such that $\gamma_i/\gamma_j \neq q^n$ for all $i \neq j$, $n \in \mathbb{Z}$. Suppose that*

$$\gamma = 1 + (\log |s|_p)/\log \max\{|s|, r\} < \Gamma = \Gamma(D)$$
$$= (2D + 1 - (1 + 4D^2)^{1/2})/(2D). \tag{2}$$

If $\varepsilon > 0$ is given, then there exists a positive constant C, depending on q, p, D, $\gamma_1, ..., \gamma_D$ and ε, such that for any rational integers $H_0, H_1, ..., H_D$, not all zero, we have

$$\left| H_0 + \sum_{j=1}^{D} H_j f(\gamma_j) \right|_p > C H^{-\theta - \varepsilon},$$

where $H = \max\{|H_0|, |H_1|, ..., |H_D|\}$, and

$$\theta = \theta(\gamma, D) = 1 + (2D - 1 + (1 + 4D^2)^{1/2})/(2 - \gamma(2D + 1 + (1 + 4D^2)^{1/2})).$$

We note that in the case $q = p^m$, $m \in \mathbb{N}$, γ has the value zero and thus (2) holds. This leads immediately to the following corollary concerning the values of the p-adic function

$$g(z) = \sum_{n=0}^{\infty} p^{an^2+bn+c} z^n, \qquad a, b, c \in \mathbb{Z}, a > 0.$$

COROLLARY. *Let $\gamma_1, ..., \gamma_D$ denote non-zero rational numbers such that $\gamma_i/\gamma_j \neq p^{2an}$ for all $i \neq j$, $n \in \mathbb{Z}$. If ε, $H_0, ..., H_D$ are as in Theorem 1, then there exists a positive constant C_1, depending on p, D, $\gamma_1, ..., \gamma_D$, a, b, c, and ε, such that*

$$\left| H_0 + \sum_{j=1}^{D} H_j g(\gamma_j) \right|_p > C_1 H^{-\theta-\varepsilon},$$

where $= \theta = \theta(D) = (2D + 1 + (1 + 4D^2)^{1/2})/2.$

If s has several prime factors p, then it may happen that (2) does not hold for any of these p. The case $D = 1$, $q = 60$ is such an example. In this situation we are only able to prove the following result on the non-existence of "s-global" relations.

THEOREM 2. *Let $\gamma_1, ..., \gamma_D$ be as in Theorem 1. Suppose that*

$$1 - (\log |s|)/\log \max\{|s|, r\} < \Gamma.$$

If $H_0, H_1, ..., H_D$ are rational integers, not all zero, then the equality

$$H_0 + H_1 f_p(\gamma_1) + \cdots + H_D f_p(\gamma_D) = 0$$

cannot hold for all $p \mid s$. Here, for each $p \mid s$, $f_p(\gamma_i)$ denotes the value $f(\gamma_i)$ of f in the corresponding $\mathbb{Q}_p$.

3. PADÉ APPROXIMATION POLYNOMIALS

Recently Stihl [8] used Maier's [5] method to find Padé approximation polynomials of the second kind for the functions $f(\gamma_1 z), ..., f(\gamma_D z)$. The considerations of Stihl can be applied also in the p-adic case if $|q|_p < 1$. Thus Satz 4 of [8] implies the following lemma (we choose there $P(x) = x$, $Q(x) \equiv 1$).

LEMMA 1. *Let $\gamma_1, ..., \gamma_D$ be rational numbers as in Theorem 1. For the parameters l, $\lambda \in \mathbb{N}$ we define the polynomial $A_{l,\lambda}$ by*

$$A_{l,\lambda}(z) = \sum_{i=0}^{Dl} z^{Dl-i} \sigma_{i,l} q^{-\binom{i+\lambda}{2}},$$

where $\sigma_{i,l}$ are defined by the identity

$$\prod_{u=1}^{D} \prod_{k=0}^{l-1} (\gamma_u - wq^{-k}) = \sum_{i=0}^{Dl} \sigma_{i,l} w^i.$$

Further, let

$$B_{l,\lambda,t}(z) = \sum_{k=0}^{Dl+\lambda-1} E_{k,t} z^k, \qquad t = 1, ..., D,$$

where the coefficients $E_{k,t}$ are given by

$$E_{k,t} = \sum_{\substack{i=0 \\ i+k-Dl \geqslant 0}}^{Dl} \sigma_{i,l} \gamma_t^{i+k-Dl} q^{\binom{i+k-Dl}{2} - \binom{i+\lambda}{2}},$$

$$k = 0, 1, ..., t = 1, ..., D.$$

Then we have, for all $t = 1, ..., D$,

$$A_{l,\lambda}(z) f(\gamma_t z) - B_{l,\lambda,t}(z) = \sum_{k=l+Dl+\lambda}^{\infty} E_{k,t} z^k = R_{l,\lambda,t}(z).$$

The proof of our results is based on certain important properties of the polynomials $A_{l,\lambda}$ and $B_{l,\lambda,t}$ which we shall consider in the following paragraph.

4. On Properties of $A_{l,\lambda}$ and $B_{l,\lambda,t}$

In our considerations we shall use, for all $a, b, c \in \mathbb{Q}$, the notation

$$(a+b, c)_0 = 1, \qquad (a+b, c)_l = \prod_{k=0}^{l-1} (a + bc^k), \qquad l = 1, 2, ...$$

From [1, p. 36], it follows that for all $l \in \mathbb{N}$

$$(a+b, c)_l = \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_c c^{\binom{k}{2}} a^{l-k} b^k, \tag{3}$$

where, for all $k = 0, 1, ..., l$,

$$\begin{bmatrix} l \\ k \end{bmatrix}_c = (1-c, c)_l / ((1-c, c)_k (1-c, c)_{l-k}).$$

We begin by proving a lemma which gives a common denominator for the coefficients of the polynomials $A_{l,\lambda}$ and $B_{l,\lambda,t'}$ and an upper estimate for the absolute values at $z=1$ of these polynomials. By $c_1, c_2, \ldots$ we shall denote effectively computable positive constants independent of l and λ.

LEMMA 2. *Let* $h=h(q)=\max\{|s|, r\}$, *and let us put* $\gamma_i=a_i/b_i$, $(a_i, b_i)=1$, $i=1, \ldots, D$. *Further, let* $\Omega_{l,\lambda}$ *denote a natural number*

$$\Omega_{l,\lambda}=s^{e(l,\lambda)}(b_1\cdots b_D)^{(D+1)l+\lambda-1},$$

where $e(l,\lambda)=[(l^2(D^2+D)+\lambda(2Dl+\lambda))/2]$. *Then all of the numbers*

$$\Omega_{l,\lambda}A_{l,\lambda}(1), \qquad \Omega_{l,\lambda}B_{l,\lambda,t}(1), \qquad t=1, \ldots, D,$$

are rational integers satisfying the inequality

$$\max\{|\Omega_{l,\lambda}A_{l,\lambda}(1)|, |\Omega_{l,\lambda}B_{l,\lambda,t}(1)|\}\leqslant c_1^{l+\lambda}h^{e(l,\lambda)}.$$

Proof. By (3) and the definition of $\sigma_{i,l}$ we obtain

$$\begin{aligned}\sum_{i=0}^{Dl}\sigma_{i,l}w^i&=\prod_{u=1}^{D}(\gamma_u-w, q^{-1})_l\\&=\prod_{u=1}^{D}\left(\sum_{k=0}^{l}\begin{bmatrix}l\\k\end{bmatrix}_{1/q}q^{-\binom{k}{2}}\gamma_u^{l-k}(-w)^k\right)\\&=\sum_{i=0}^{Dl}(-1)^i\sum_{\substack{\nu_1+\cdots+\nu_D=i\\0\leqslant\nu_j\leqslant l}}\\&\quad\times\left\{\begin{bmatrix}l\\\nu_l\end{bmatrix}_{1/q}\cdots\begin{bmatrix}l\\\nu_D\end{bmatrix}_{1/q}q^{-\binom{\nu_1}{2}-\cdots-\binom{\nu_D}{2}}\gamma_1^{l-\nu_1}\cdots\gamma_D^{l-\nu_D}w^i\right\}.\end{aligned}$$

Since

$$\begin{bmatrix}l\\\nu\end{bmatrix}_{1/q}q^{-\binom{\nu}{2}}=q^{-\nu l}\begin{bmatrix}l\\\nu\end{bmatrix}_q q^{\binom{\nu+1}{2}},$$

this yields

$$\begin{aligned}\sigma_{i,l}&=(-1)^iq^{-il}\sum_{\substack{\nu_1+\cdots+\nu_D=i\\0\leqslant\nu_j\leqslant l}}\\&\quad\times\left\{\begin{bmatrix}l\\\nu_1\end{bmatrix}_q\cdots\begin{bmatrix}l\\\nu_D\end{bmatrix}_q q^{\binom{\nu_1+1}{2}+\cdots+\binom{\nu_D+1}{2}}\gamma_1^{l-\nu_1}\cdots\gamma_D^{l-\nu_D}\right\}.\end{aligned}$$

By Hilfssatz 4 of [8] it follows that

$$r^{\nu(l-\nu)}\begin{bmatrix}l\\\nu\end{bmatrix}_q\in\mathbb{Z}, \qquad \nu=0, 1, \ldots, l.$$

If $\nu_1 + \cdots + \nu_D = i$, then we have

$$q^{-il+\binom{\nu_1+1}{2}+\cdots+\binom{\nu_D+1}{2}-\binom{i+\lambda}{2}} r^{-\nu_1(l-\nu_1)-\cdots-\nu_D(l-\nu_D)} = s^{-s(i,l)} r^{r(i,l)},$$

where

$$s(i,l) = il + \binom{i+\lambda}{2} - \left(\sum_{j=1}^{D} \nu_j^2 + i\right)\Big/2,$$

$$r(i,l) = \binom{i+\lambda}{2} + \left(\sum_{j=1}^{D} \nu_j^2 - i\right)\Big/2.$$

Since, for all $0 \leqslant i \leqslant Dl$, $0 \leqslant k \leqslant Dl + \lambda - 1$, $i + k - Dl \geqslant 0$,

$$s(i,l) - \binom{i+k-Dl}{2} \leqslant s(i,l) \leqslant i(l-1/2) + i^2(D-1)/(2D) + \lambda(2i+\lambda)/2 \leqslant e(l,\lambda),$$

and

$$r(i,l) - \binom{i+k-Dl}{2} \geqslant \binom{i+\lambda}{2} + \left(\sum_{j=1}^{D} \nu_j^2 - i\right)\Big/2 - \binom{i+\lambda-1}{2} > 0,$$

it follows immediately that all the numbers $\Omega_{l,\lambda} A_{l,\lambda}(1)$ and $\Omega_{l,\lambda} B_{l,\lambda,t}(1)$ are integers.

We obviously have

$$\left| \begin{bmatrix} l \\ \nu \end{bmatrix}_q \right| \leqslant c_2^l \, |q|^{\delta\nu(l-\nu)},$$

where $\delta = 0$, if $|q| < 1$, and $\delta = 1$, if $|q| > 1$. In the case $|q| < 1$ we thus obtain, for all $0 \leqslant i \leqslant Dl$, $0 \leqslant k \leqslant Dl + \lambda - 1$, $i + k - Dl \geqslant 0$,

$$\left|\sigma_{i,l} q^{\binom{i+k-Dl}{2} - \binom{i+\lambda}{2}}\right| \leqslant c_3^{l+\lambda} \, |q^{-1}|^{il + \binom{i+\lambda}{2} - (\min \Sigma + i)/2} \leqslant c_3^{l+\lambda} \, |q|^{-e(l,\lambda)},$$

where we have used the notation

$$\Sigma = \nu_1^2 + \cdots + \nu_D^2, \qquad \nu_1 + \cdots + \nu_D = i.$$

On the other hand, if $|q| > 1$,

$$\left|\sigma_{i,l} q^{\binom{i+k-Dl}{2} - \binom{i+\lambda}{2}}\right| \leqslant c_4^{l+\lambda} \, |q|^{-il + \binom{i+\lambda-1}{2} - \binom{i+\lambda}{2}} \max\{(\Sigma+i)/2 + il - \Sigma\} \leqslant c_5^{l+\lambda} \, |q|^{(i - \min \Sigma)/2} \leqslant c_5^{l+\lambda}.$$

These considerations imply the inequality

$$\max\{|\Omega_{l,\lambda}A_{l,\lambda}(1)|, |\Omega_{l,\lambda}B_{l,\lambda,t}(1)|\} \leqslant c_1^{l+\lambda}h^{e(l,\lambda)},$$

which gives Lemma 2.

Next we shall consider the determinant

$$\Delta_{l,\bar{\lambda}}(z) = \begin{vmatrix} A_{l,\lambda(0)}(z) & B_{l,\lambda(0),1}(z) & \cdots & B_{l,\lambda(0),D}(z) \\ A_{l,\lambda(1)}(z) & B_{l,\lambda(1),1}(z) & \cdots & B_{l,\lambda(1),D}(z) \\ \vdots & \vdots & & \vdots \\ A_{l,\lambda(D)}(z) & B_{l,\lambda(D),1}(z) & \cdots & B_{l,\lambda(D),D}(z) \end{vmatrix}.$$

Our choice of Δ is different from the corresponding choice of Stihl [8], and it implies the following useful lemma.

LEMMA 3. *If* $\lambda(j) = \lambda + j$, $j = 0, 1, ..., D$, *then*

$$\Delta_{l,\bar{\lambda}}(z) = Kz^{(D^2+D)(l+1/2)+D\lambda},$$

where $K \neq 0$ *is a rational constant.*

Proof. Our $\Delta_{l,\bar{\lambda}}$ is a polynomial of degree

$$\leqslant Dl + \sum_{j=1}^{D} (Dl + \lambda + j - 1) = (D^2 + D)l + D\lambda + \binom{D}{2}.$$

The coefficient of $z^{(D^2+D)(l+1/2)+D\lambda}$ is the determinant K,

$$K = \sigma_{0,l}q^{-\binom{\lambda}{2}} \begin{vmatrix} E_{Dl+\lambda(1)-1,1} & \cdots & E_{Dl+\lambda(1)-1,D} \\ E_{Dl+\lambda(2)-1,1} & \cdots & E_{Dl+\lambda(2)-1,D} \\ \vdots & & \vdots \\ E_{Dl+\lambda(D)-1,1} & \cdots & E_{Dl+\lambda(D)-1,D} \end{vmatrix},$$

where, by Lemma 1,

$$\begin{aligned} E_{Dl+\lambda(j)-1,t} &= \sum_{i=0}^{Dl} \sigma_{i,l}\gamma_t^{i+\lambda(j)-1} q^{\binom{i+\lambda(j)-1}{2} - \binom{i-\lambda(j)}{2}} \\ &= \sum_{i=0}^{Dl} \sigma_{i,l}(\gamma_t/q)^{i+\lambda(j)-1} \\ &= (\gamma_t/q)^{\lambda(j)-1} \sum_{i=0}^{Dl} \sigma_{i,l}(\gamma_t/q)^i \\ &= (\gamma_t/q)^{\lambda(j)-1} \prod_{u=1}^{D} (\gamma_u - \gamma_t/q, q^{-1})_l, \qquad j, t = 1, ..., D. \end{aligned}$$

Therefore

$$K = (\gamma_1 \cdots \gamma_D)^{l+\lambda} q^{-\binom{\lambda}{2} - D\lambda} \prod_{t=1}^{D} \prod_{u=1}^{D} (\gamma_u - \gamma_t/q, q^{-1})_l$$

$$\times \begin{vmatrix} 1 & \cdots & 1 \\ \gamma_1/q & \cdots & \gamma_D/q \\ \vdots & & \vdots \\ (\gamma_1/q)^{D-1} & \cdots & (\gamma_D/q)^{D-1} \end{vmatrix},$$

and since $\gamma_u \neq \gamma_t q^n$ for all $u \neq t$, $n \in \mathbb{Z}$, we have $K \neq 0$.

On the other hand, since $A_{l,\lambda}(z) f(\gamma_t z) - B_{l,\lambda,t}(z) = R_{l,\lambda,t}(z)$ (see Lemma 1), $\Delta_{l,\bar{\lambda}}$ can be expressed in the form

$$\Delta_{l,\bar{\lambda}}(z) = \pm \begin{vmatrix} A_{l,\lambda(0)}(z) & R_{l,\lambda(0),1}(z) & \cdots & R_{l,\lambda(0),D}(z) \\ A_{l,\lambda(1)}(z) & R_{l,\lambda(1),1}(z) & \cdots & R_{l,\lambda(1),D}(z) \\ \vdots & \vdots & & \vdots \\ A_{l,\lambda(D)}(z) & R_{l,\lambda(D),1}(z) & \cdots & R_{l,\lambda(D),D}(z) \end{vmatrix}.$$

Thus $\Delta_{l,\bar{\lambda}}$ has at $z = 0$ a zero of order at least

$$\sum_{j=0}^{D-1} (1 + Dl + \lambda + j) = (D^2 + D)l + D\lambda + \binom{D}{2}.$$

Together with the above considerations this implies Lemma 3.

In the following Section 5 we shall consider more closely the properties of the above used functions $R_{l,\lambda,t}$.

5. The Estimation of $R_{l,\lambda,t}$

We also need an upper estimate for the p-value of $\Omega_{l,\lambda} R_{l,\lambda,t}(1)$, $t = 1, ..., D$. This is achieved in the following lemma.

Lemma 4. *If $p \mid s$, then the inequalities*

$$|\Omega_{l,\lambda} R_{l,\lambda,t}(1)|_p \leqslant c_6^{l+\lambda} |s|_p^{e(l,\lambda) + (l^2 + 2l\lambda)/2}, \qquad t = 1, ..., D,$$

are valid for all $l \geqslant c_7$.

Proof. We have, by Lemma 1,

$$
\begin{aligned}
|R_{l,\lambda,t}(1)|_p &\leqslant \max_{k \geqslant l+Dl+\lambda} \{|E_{k,t}|_p\} \\
&\leqslant \max_{\substack{k \geqslant l+Dl+\lambda \\ 0 \leqslant i \leqslant Dl}} \left\{\left|\sigma_{i,l}\gamma_t^{i+k-Dl} q^{\binom{i+k-Dl}{2}-\binom{i+\lambda}{2}}\right|_p\right\} \\
&\leqslant \max_{0 \leqslant i \leqslant Dl} \left\{\left|\sigma_{i,l}\gamma_t^{i+l+\lambda} q^{\binom{i+l+\lambda}{2}-\binom{i+\lambda}{2}}\right|_p\right\} \\
&\quad \times \max_{\substack{k \geqslant l+Dl+\lambda \\ 0 \leqslant i \leqslant Dl}} \left\{\left|\gamma_t^{k-(l+Dl+\lambda)} q^{\binom{i+k-Dl}{2}-\binom{i+l+\lambda}{2}}\right|_p\right\}.
\end{aligned}
$$

Since

$$\binom{i+k-Dl}{2}-\binom{i+l+\lambda}{2}=\sum_{j=i+l+\lambda}^{i+k-Dl-1} j,$$

and $|\gamma_t q^j|_p < 1$, $t=1, ..., D$, for all $j \geqslant c_8$, the last term in the above product is <1 for all $l \geqslant c_7$. Therefore

$$
\begin{aligned}
|R_{l,\lambda,t}(1)|_p &\leqslant c_9^{l+\lambda} \max_{0 \leqslant i \leqslant Dl} \left\{|q|_p^{-il+\min(\Sigma+i)/2} \, |q|_p^{\binom{i+l+\lambda}{2}-\binom{i+\lambda}{2}}\right\} \\
&\leqslant c_9^{l\lambda} \max_{0 \leqslant i \leqslant Dl} \left\{|q|_p^{(l^2+2l\lambda+\min(\Sigma+i)-l)/2}\right\} \\
&\leqslant c_{10}^{l+\lambda} |s|_p^{(l^2+2l\lambda)/2}
\end{aligned}
$$

for all $l \geqslant c_7$. By using the definition of $\Omega_{l,\lambda}$ we now immediately obtain Lemma 4.

6. The Proof of Theorem 1 and Its Corollary

Let us put

$$L = H_0 + \sum_{t=1}^{D} H_t f(\gamma_t),$$

where H_i are integers, not all zero, and $\max\{|H_i|\} \leqslant H$. Then we immediately have

$$
\begin{aligned}
\Omega_{l,\lambda} A_{l,\lambda}(1) m &= \Omega_{l,\lambda}\Big(A_{l,\lambda}(1) H_0 + \sum_{t=1}^{D} H_t (B_{l,\lambda,t}(1) + R_{l,\lambda,t}(1))\Big) \\
&= \Omega_{l,\lambda}\Big(A_{l,\lambda}(1) H_0 + \sum_{t=1}^{D} B_{l,\lambda,t}(1) H_t\Big) \\
&\quad + \sum_{t=1}^{D} H_t \Omega_{l,\lambda} R_{l,\lambda,t}(1). \qquad (4)
\end{aligned}
$$

Let us choose for λ the value $[\beta l]$, where $\beta > 0$ is a constant that will be fixed later. Then, by Lemma 3, there exists a $\lambda(j) = \lambda + j$, $0 \leqslant j \leqslant D$, such that the number

$$Y = \Omega_{l,\lambda(j)}(Z_{l,\lambda(j)} H_0 + \sum_{t=1}^{D} B_{l,\lambda(j),t}(1) H_t)$$

is different from zero. Further, by Lemma 2, $Y \in \mathbb{Z}$. Thus

$$1 \leqslant |Y|\, |Y|_p.$$

By using again Lemma 2 we obtain an estimate

$$|Y| \leqslant H c_{11}^{l+\lambda(j)} h^{e(l,\lambda(j))},$$

which implies an inequality

$$\begin{aligned} 1 &\leqslant H c_{11}^{l+\lambda(j)} h^{e(l,\lambda(j)} |Y|_p \\ &\leqslant H c_{11}^{l+\lambda(j)} h^{e(l,\lambda(j))} \\ &\quad \times \max \left\{ |\Omega_{l,\lambda(j)} A_{l,\lambda(j)}(1) L|_{p'} \left| \sum_{t=1}^{D} H_t \Omega_{l,\lambda(j)} R_{l,\lambda(j),t}(1) \right|_p \right\}. \end{aligned} \tag{5}$$

From Lemma 4 it now follows that for all $l \geqslant c_7$

$$\left| \sum_{t=1}^{D} H_t \Omega_{l,\lambda(j)} R_{l,\lambda(j),t}(1) \right|_p \leqslant c_6^{l+\lambda(j)} |s|_p^{e(l,\lambda(j)) + (l^2 + 2l\lambda(j))/2}.$$

Therefore, let us consider the expression

$$X = H(c_6 c_{11})^{l+\lambda(j)} h^{e(l,\lambda(j))} |s|_p^{e(l,\lambda(j)) + (l^2 + 2l\lambda(j))/2}.$$

Since $\lambda = (\beta l]$, we have

$$X \leqslant H c_{12}^{l} (h^{1-\Gamma(\beta)} |s|_p)^{\Gamma_1(\beta) l^2/2},$$

where $\Gamma_1(\beta) = \beta^2 + 2(D+1)\beta + D^2 + D + 1$, $\Gamma(\beta) = (1+2\beta)/\Gamma_1(\beta)$. The choice $\beta = \bar{\beta} = (-1 + (1+4D^2)^{1/2})/2$ gives a maximal value

$$\Gamma = (2D + 1 - (1+4D^2)^{1/2})/(2D)$$

for $\Gamma(\beta)$. Let us denote

$$R = (h^{1-\Gamma} |s|_p)^{\Gamma_1/2}, \qquad \Gamma_1 = \Gamma_1(\bar{\beta}).$$

Then $R < 1$, since $h^{1-\Gamma} |s|_p < 1$ by our assumption (2). With a given $\varepsilon > 0$ we now choose

$$l = [((1+\varepsilon)(\log H)/\log R^{-1})^{1/2}] + 1.$$

Then $X < 1$ for all $H \geqslant c_{13}$. Here we suppose c_{13} is large enough to imply $l \geqslant c_7$. By (5) it therefore follows that

$$1 \leqslant Hc_{11}^{l+\lambda(j)} h^{e(l,\lambda(j))} |\Omega_{l,\lambda(j)} A_{l,\lambda(j)}(1) L|_p. \tag{6}$$

Lemma 2 says that $\Omega_{l,\lambda(j)} A_{l,\lambda(j)}(1) \in \mathbb{Z}$. Thus (2) and (6) give the estimates

$$\begin{aligned} 1 &\leqslant |L|_p Hc_{14}^l h^{\Gamma_1(1-\Gamma)l^2/2} \\ &\leqslant |L|_p Hc_{15}^{(\log H)^{1/2}} H^{(1+\varepsilon)\Gamma_1(1-\Gamma)(\log h)/(-2\log R)} \\ &\leqslant |L|_p c_{15}^{(\log H)^{1/2}} H^{1-(1+\varepsilon)(1-\Gamma)(\log h)/((1-\Gamma)\log h + \log |s|_p)} \\ &< |L|_p H^{1+(1+2\varepsilon)(1-\Gamma)/(\Gamma-\gamma)}, \end{aligned}$$

for all $H \geqslant c_{16}$. Here $(1-\Gamma)/(\Gamma-\gamma) = \theta(\gamma, D) - 1 = \theta - 1$. Thus we have

$$|L|_p > H^{-\theta - 2\varepsilon(\theta-1)}$$

for all $H \geqslant c_{16}$. This proves Theorem 1.

To prove the corollary it is enough to note that by the choice $q = p^{2a}$ we have

$$g(z) = p^c f(p^{b-a} z).$$

7. The Proof of Theorem 2

Suppose that

$$L_p = H_0 + \sum_{t=1}^{D} H_t f_p(\gamma_t) = 0$$

for all $p \mid s$. As in the beginning of the above proof of Theorem 1 we find again a rational integer

$$Y = \Omega_{l,\lambda(j)}(A_{l,\lambda(j)}(1) H_0 + \sum_{t=1}^{D} B_{l,\lambda(j),t}(1) H_t) \neq 0.$$

Thus

$$1 = |Y| \prod_p |Y|_p \leqslant |Y| \prod_{p \mid s} |Y|_p,$$

where, as before

$$|Y| \leqslant Hc_{11}^{l+\lambda(j)} h^{e(l,\lambda(j))}.$$

We supposed above that $L_p = 0$ for all $p \mid s$. By (4) it thus follows that

$$|Y|_p = \left| \sum_{t=1}^{D} H_t \Omega_{l,\lambda(j)} R_{l,\lambda(j),t}(1) \right|_p$$

for all $p \mid s$. Therefore Lemma 4 implies an inequality

$$|Y|_p \leqslant c_{17}^{l+\lambda(j)} |s|_p^{e(l,\lambda(j)) + (l^2 + 2l\lambda(j))/2}$$

for all $p \mid s$, $l \geqslant c_7$. By choosing λ similar to as in the above proof we now obtain

$$1 \leqslant Hc_{18}^{l} \left(h^{1-\Gamma} \prod_{p \mid s} |s|_p \right)^{\Gamma_1 l^2/2}. \tag{7}$$

By the assumption of Theorem 2 we have $(1-\Gamma)\log h < \log |s|$, which implies an inequality $h^{1-\Gamma}/|s| < 1$. Thus (7) leads to a contradiction for all sufficiently large l. This proves Theorem 2.

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